CHAPTER 3: METHODS OF CIRCUIT ANALYSIS

3.1 Nodal Analysis

- Provides a general procedure for analyzing circuits using node voltages as the circuit viarables.
- We assume that circuits do not contain voltage sources.
- We are interested in finding the node voltages.
- Given a circuit with *n* nodes without voltage sources, the nodal analysis of the circuit involves the following steps:
  1. Select a node as the reference node. Assign voltages $v_1, v_2, \ldots, v_{n-1}$ to the remaining *n-1* nodes. The voltages are referenced with respect to the reference node.
  2. Apply KCL to each of the *n-1* nonrefernce nodes. Use Ohm's law to express the branch currents in term of node voltages.
  3. Solve the resulting simultaneous equations to obtain the unknown node voltages.
- For example:
  Consider the circuit in Figure 3.1:

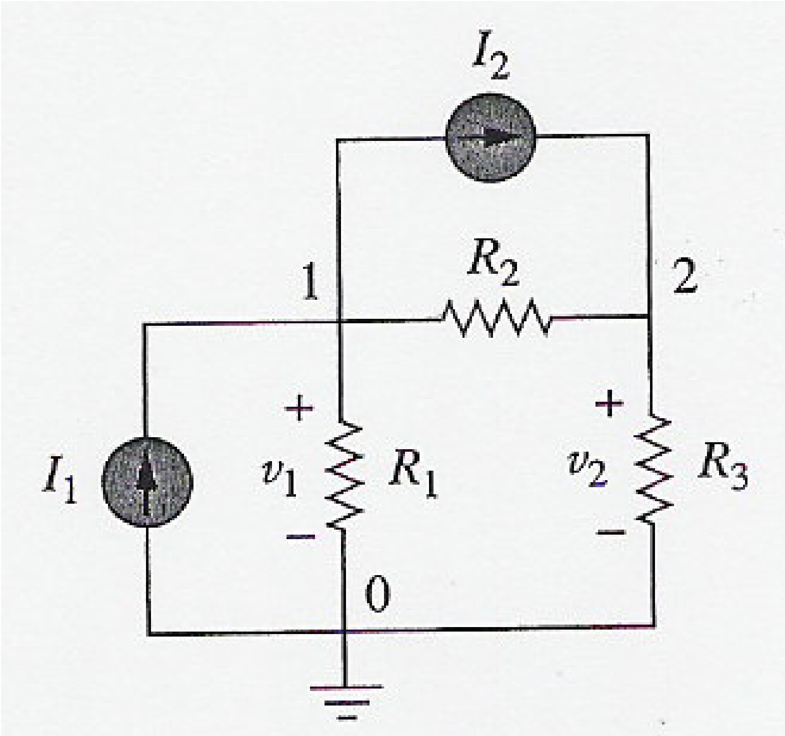

Figure 3.1

First step:

- Select a node as the reference node.
- The reference node is commonly called the ground since it is assumed to have zero potential.
- From the circuit, node 0 is the reference node.
- Nodes 1 and 2 are assigned voltages v_1 and v_2 respectively.

Second step:

- Apply KCL to each nonreference node in the circuit.
- Thus, we redraw the circuit

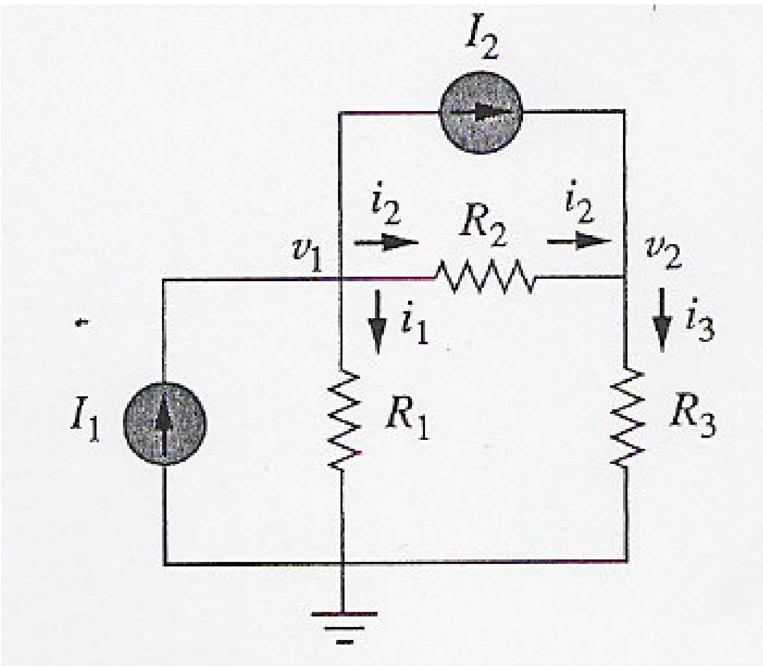

Figure 3.2

- At node 1,

$$I_1 = I_2 + i_1 + i_2$$

- At node 2,

$$I_2 + i_2 = i_3$$

- Appliy Ohm's law to express the unknown currents i_1, i_2 and i_3 in terms of node voltages.

  Note: Since the resistance is a passive element, current must always flow from a higher potential to a lower potential.

- Thus,

$$i_1 = \frac{v_1 - 0}{R_1} \qquad or \qquad i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2} \qquad or \qquad i_2 = G_2(v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3} \qquad or \qquad i_3 = G_3 v_2$$

- Thus,

$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

In terms of conductances,

$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2)$$
$$I_2 + G_2(v_1 - v_2) = G_3 v_2$$

Third step:

Based on the equation obtained, find the value for the node voltages.

Two methods can be used:

(a) Elimination method

(b) Cramer's rule

- Example:

Obtain the node voltages in the circuit in Figure 3.3:

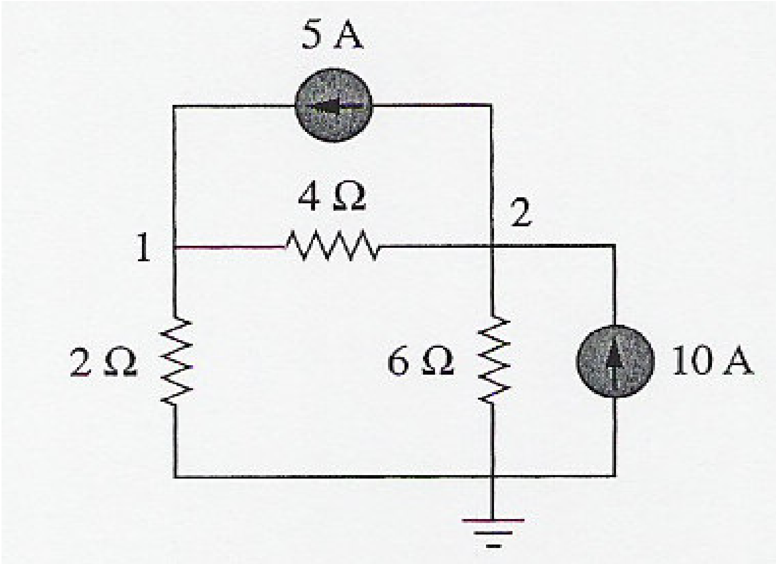

Figure 3.3

Redraw the circuit as shown in Figure 3.4:

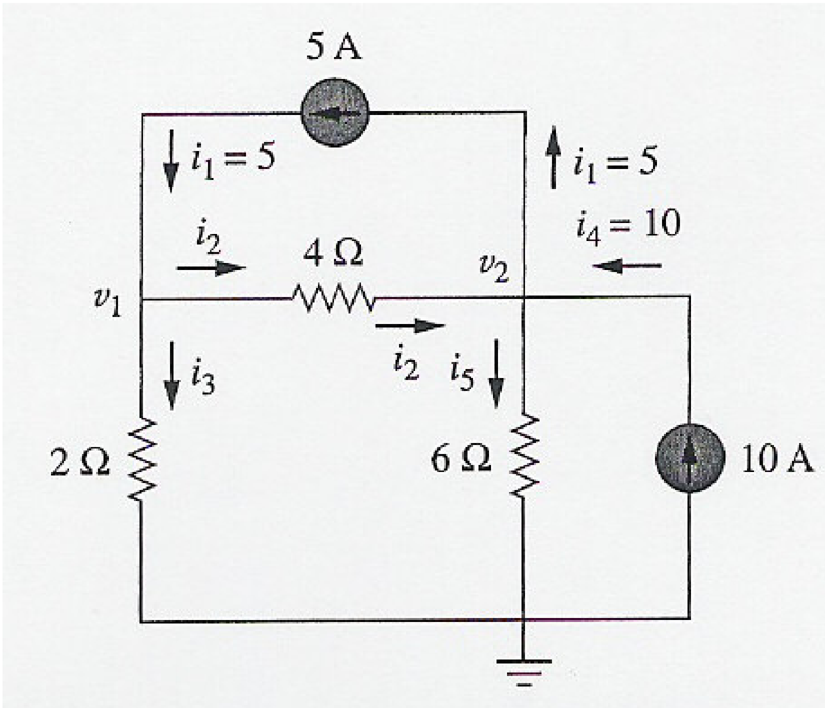

Figure 3.4

At node 1, applying KCL and Ohm's Law:

$$i_1 = i_2 + i_3$$

$$5 = \frac{(v_1 - v_2)}{4} + \frac{(v_1 - 0)}{2}$$

$$20 = v_1 - v_2 + 2v_1$$

$$3v_1 - v_2 = 20 \qquad \text{--------(a)}$$

At node 2, applying KCL and Ohm's Law:

$$i_2 + i_4 = i_1 + i_5$$

$$\frac{(v_1 - v_2)}{4} + 10 = 5 + \frac{(v_2 - 0)}{6}$$

$$3v_1 - 3v_2 + 120 = 60 + 2v_2$$

$$-3v_1 + 5v_2 = 60 \qquad \text{--------(b)}$$

To find the node voltages,

Method 1 – elimination technique:

From (a) and (b), we obtain:

$$4v_2 = 80 \Rightarrow v_2 = 20\text{V}$$

$$\therefore v_1 = 13.33\text{V}$$

Method 2 – Cramer's rule:

We need to put equations (a) and (b) in matrix form

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

The determinant of the matrix is

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$

$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.33\text{V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20\text{V}$$

Thus,

$$i_1 = 5\text{A} \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6667\text{A}$$

$$i_3 = \frac{v_1}{2} = 6.666\text{A} \qquad i_4 = 10\text{A}$$

$$i_5 = \frac{v_2}{6} = 3.333\text{A}$$

- Example on Cramer's rule for 3 viarables:

  Find the value of v_1, v_2 and v_3

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$$a = \frac{\Delta_1}{\Delta}, \qquad b = \frac{\Delta_2}{\Delta}, \qquad c = \frac{\Delta_3}{\Delta}$$

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \\ 3 & -2 & -1 \\ -4 & 7 & -1 \end{vmatrix}$$

$$\Delta = 21 - 12 + 4 + 14 - 9 - 8 = 10$$

$$\Delta_1 = \begin{vmatrix} 12 & -2 & -1 \\ 0 & 7 & -1 \\ 0 & -3 & 1 \\ 12 & -2 & -1 \\ 0 & 7 & -1 \end{vmatrix} = 84 + 0 + 0 - 0 - 36 - 0 = 48$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 & -1 \\ -4 & 0 & -1 \\ 2 & 0 & 1 \\ 3 & 12 & -1 \\ -4 & 0 & -1 \end{vmatrix} = 0 + 0 - 24 - 0 - 0 + 48 = 24$$

$$\Delta_3 = \begin{vmatrix} 3 & -2 & 12 \\ -4 & 7 & 0 \\ 2 & -3 & 0 \\ 3 & -2 & 12 \\ -4 & 7 & 0 \end{vmatrix} = 0 + 144 + 0 - 168 - 0 - 0 = -24$$

Thus,

$$a = 4.8, \qquad b = 2.4, \qquad c = -2.4$$

3.2 Nodal Analysis with Voltage Sources.

- Consider the circuit in Figure 3.5:

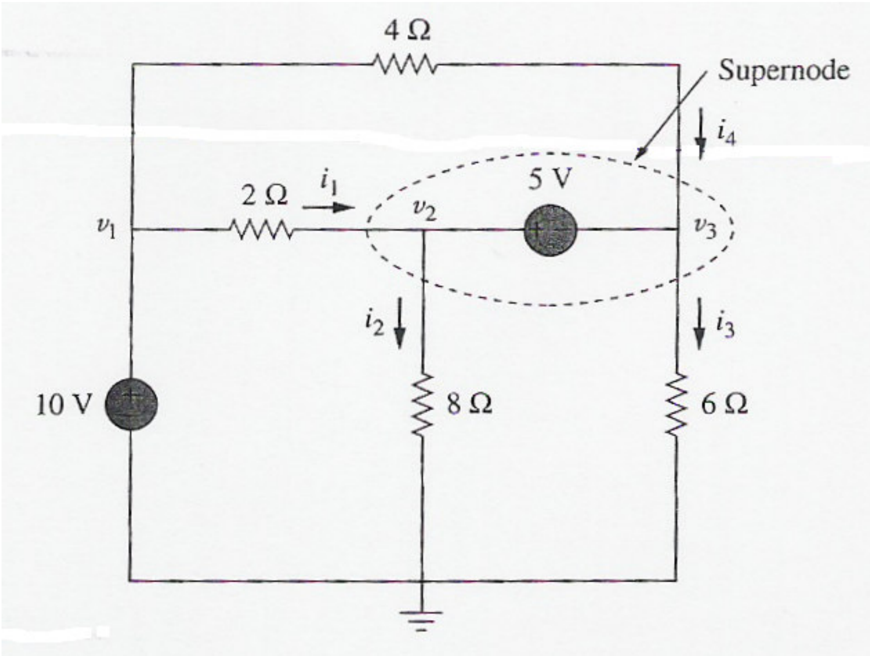

Figure 3.5

- **Case 1**

  If a voltage source is connected between the reference node and a nonreference node – set the voltage at the nonreference node equal to the voltage of the source,

$$v_1 = 10\text{V}$$

- **Case 2**

  If the voltage source is connected between two nonreference nodes, the two nonreference nodes form a generalized node or supernode – apply both KCL and KVL to determine the node voltages.

For v_2 and v_3:

Apply KCL,

$$i_1 + i_4 = i_2 + i_3$$

or

$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$

Apply KVL,

$$-v_2 + 5 + v_3 = 0$$

$$v_2 - v_3 = 5$$

Then, based on equations, find v_2 and v_3

- Example 1:

Find v and i in the circuit as shown in Figure 3.6:

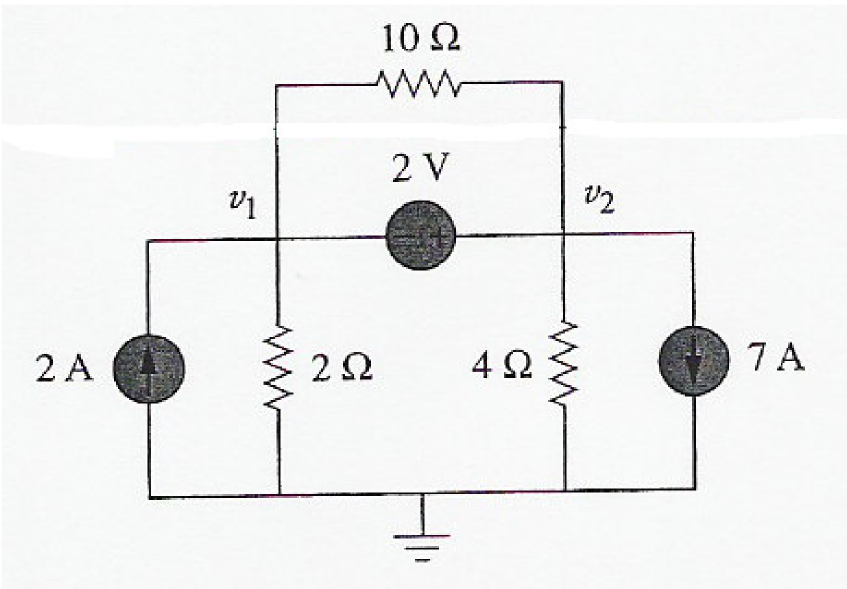

Figure 3.6

The supernode contains the 2V source, nodes 1 and 2, and the 10Ω resistor.

Redraw the circuit,

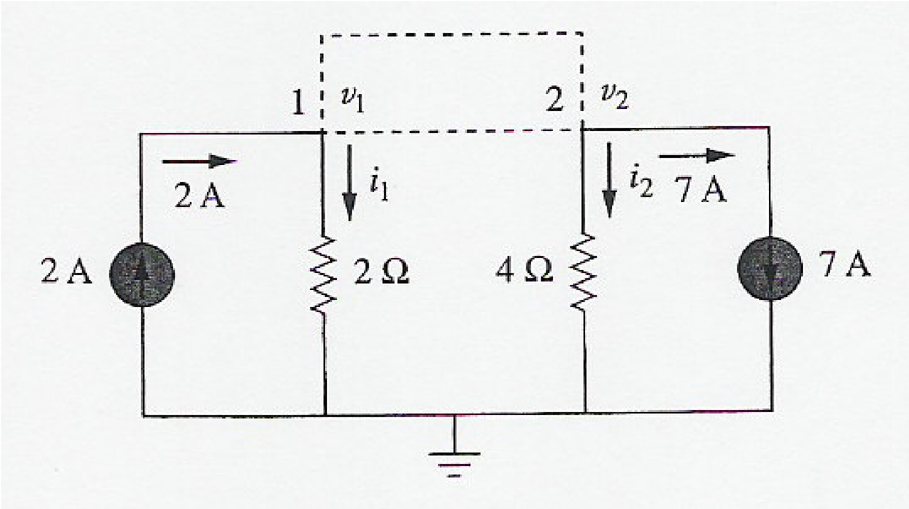

Figure 3.7

Applying KCL to the supernode,

$$2 = i_1 + i_2 + 7$$

$$2 = \frac{v_1 - 0}{2} + \frac{v_2 - 0}{4} + 7$$

$$8 = 2v_1 + v_2 + 28$$

$$v_2 = -20 - 2v_1 \qquad \text{--------(c)}$$

Redraw again the circuit as shown below:

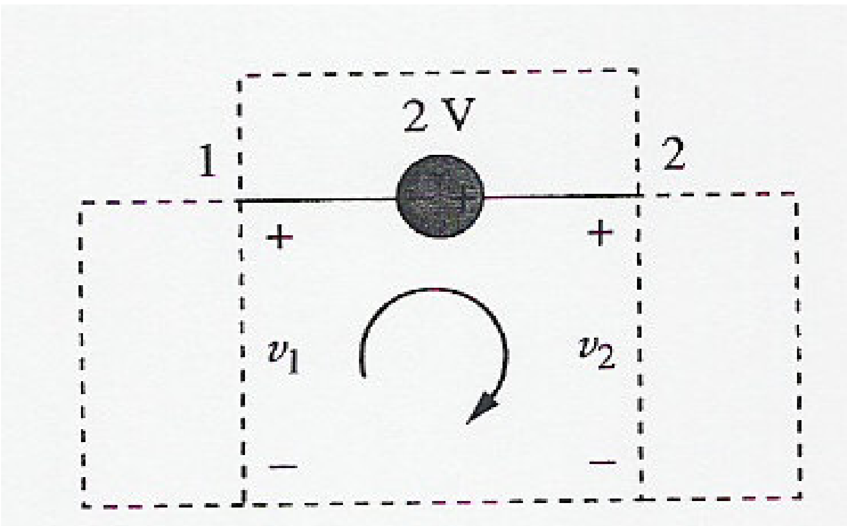

Figure 3.8

Applying KVL,

$$-v_1 - 2 + v_2 = 0$$

$$v_2 = v_1 + 2 \qquad \text{--------(d)}$$

From (c) and (d),

$$v_2 = v_1 + 2 = -20 - 2v_1$$

$$3v_1 = -22\text{V}$$

$$v_1 = -7.333\text{V}$$

$$v_2 = v_1 + 2 = -5.333\text{V}$$

Note: the 10Ω resistor does not make any difference because it is connected across the supernode.

- Example 2:

  Find the node voltage in the circuit of Figure 3.9

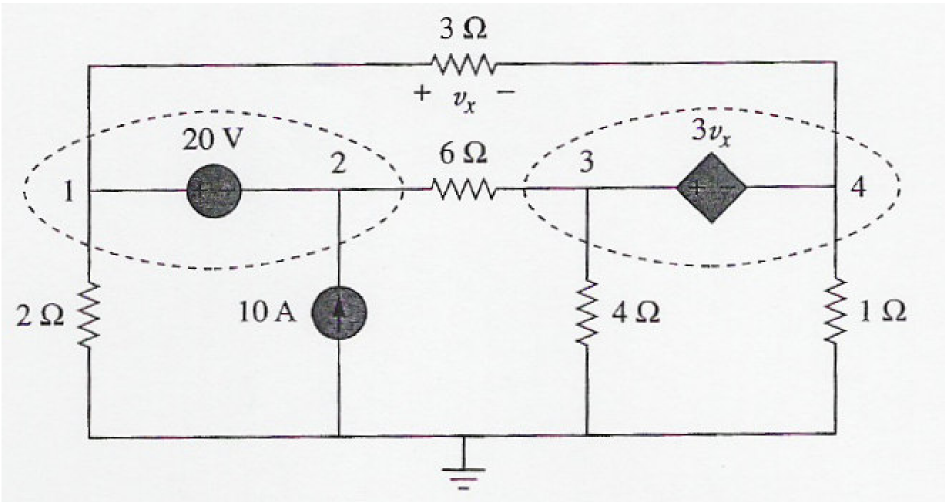

Figure 3.9

Nodes 1 and 2 & nodes 3 and 4 form supernode. Applying KCL to the two supernodes, we get:

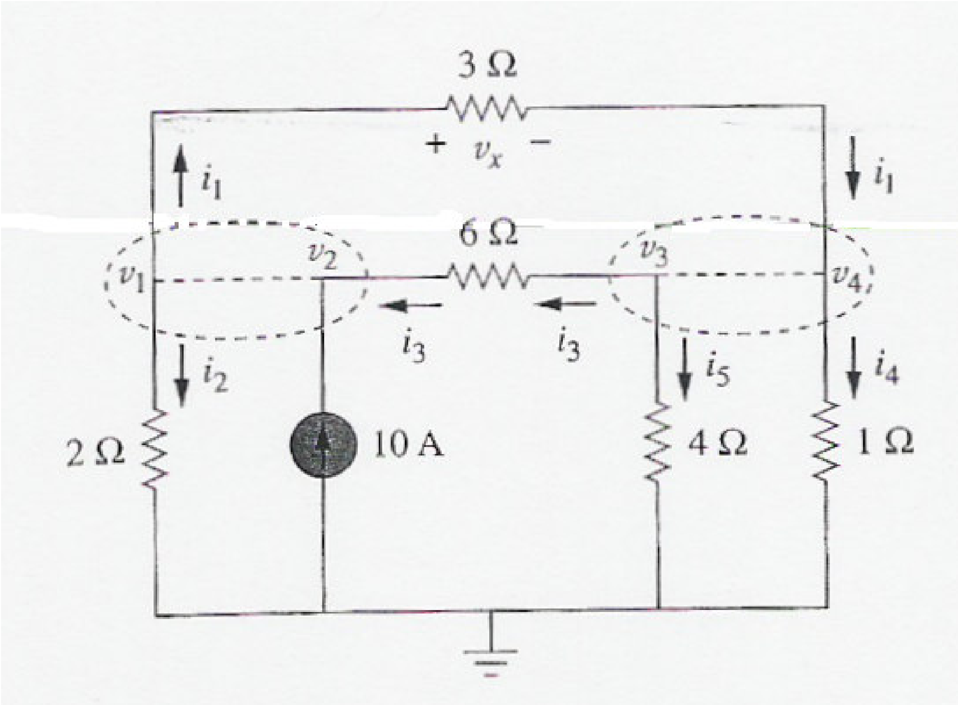

Figure 3.10

For supernode 1-2,

$$i_3 + 10 = i_1 + i_2$$

$$\frac{v_3 - v_2}{6} + 10 = \frac{v_1 - v_4}{3} + \frac{v_2}{2}$$

$$5v_1 + v_2 - v_3 - 2v_4 = 60 \qquad \text{----(a)}$$

For supernode 3-4,

$$i_1 = i_3 + i_4 + i_5$$

$$\frac{v_1 - v_4}{3} = \frac{v_3 - v_2}{6} + \frac{v_4}{1} + \frac{v_3}{4}$$

$$4v_1 + 2v_2 - 5v_3 - 16v_4 = 0 \qquad \text{----(b)}$$

Apply KVL to the branches involving the voltage sources as shown in Figure 3.11.

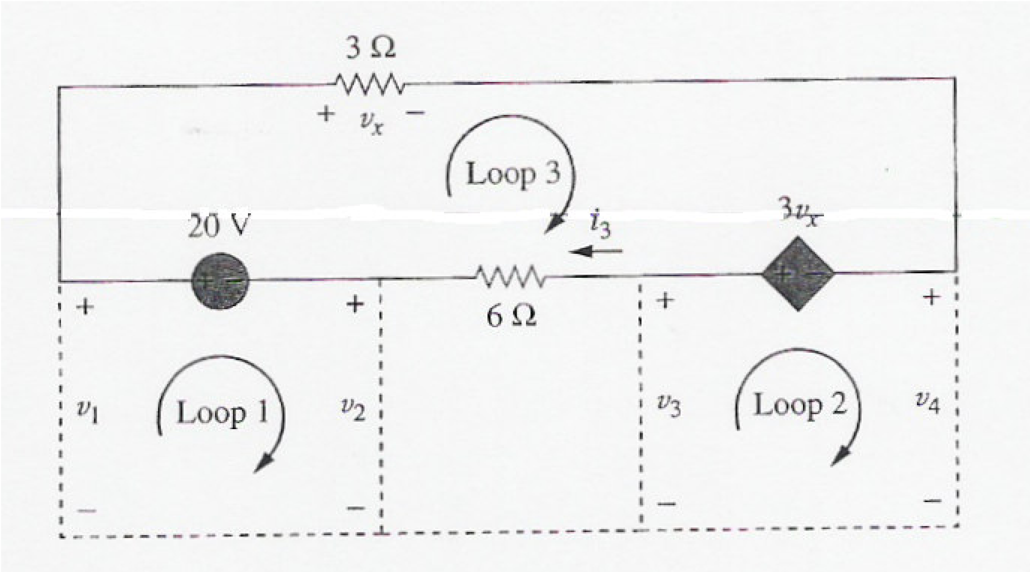

Figure 3.11

For loop 1,

$$-v_1 + 20 + v_2 = 0$$

$$v_1 - v_2 = 20 \qquad \text{----(c)}$$

For loop 2,

$$-v_3 + 2v_x + v_4 = 0$$

But,

$$v_x = v_1 - v_4$$

So that,

$$3v_1 - v_3 - 2v_4 = 0 \qquad \text{----(d)}$$

For loop 3,

$$v_x - 3v_x + 6i_3 - 20 = 0$$

But,

$$6i_3 = v_3 - v_2 \text{ and } v_x = v_1 - v_4$$

So that,

$$-2v_1 - v_2 + v_3 + 2v_4 = 20 \qquad \text{----(e)}$$

Substituting equation (c) into (a) and (b), gives,

$$6v_1 - v_3 - 2v_4 = 80 \qquad \text{----(f)}$$

$$6v_1 - 5v_3 - 16v_4 = 40 \qquad \text{----(g)}$$

Equation (d), (f) and (g) in matrix form,

$$\begin{bmatrix} 3 & -1 & -2 \\ 6 & -1 & -2 \\ 6 & -5 & -16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 80 \\ 40 \end{bmatrix}$$

Using Cramer's rule,

$$v_1 = 26.667\,\text{V}, \qquad v_3 = 173.333\,\text{V}$$

$$v_4 = -46.667\,\text{V}$$

Thus,

$$v_2 = 6.667\,\text{V}$$

3.3 Mesh Analysis

- Mesh analysis is another procedure for analyzing a circuit using mesh currents as the circuit viarables.

A **mesh** is a loop which does not contain other loop within it.

- Consider the circuit as shown in Figure 3.9:

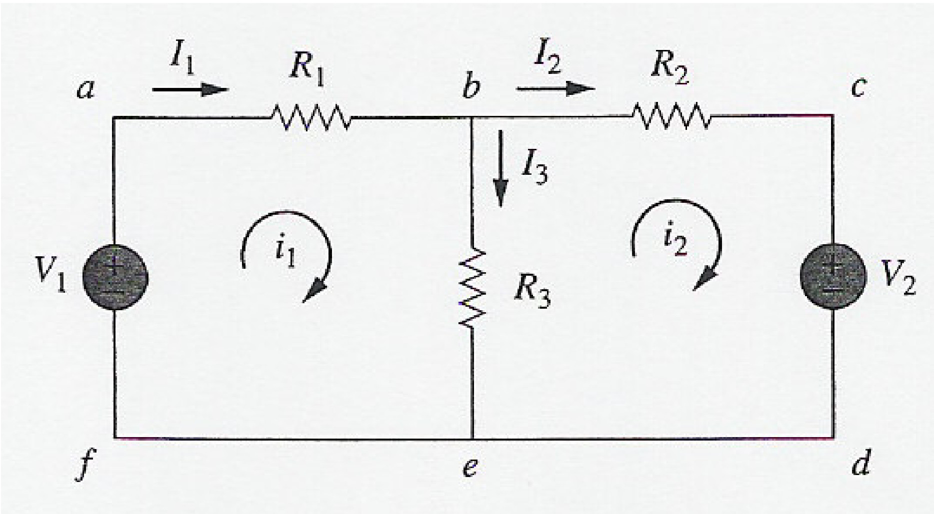

Figure 3.12

- Paths abefa and bcdeb are meshes but path abcdefa is not a mesh.
- The current through a mesh is known as mesh current. In mesh analysis, we apply KVL to find the mesh currents in a given circuit.
- Steps to determine mesh currents:
  (i) Determine the number of mesh in circuit. Assign mesh currents to the meshes (refer to Figure 3.12).

(ii) Apply KVL to each mesh (in term of mesh current)

Mesh 1:

$$-V_1 + R_1 i_1 + R_3(i_1 - i_2) = 0$$

$$(R_1 + R_3)i_1 - R_3 i_2 = V_1$$

Mesh 2:

$$R_2 i_2 + V_2 + R_3(i_2 - i_1) = 0$$

$$-R_3 i_1 + (R_2 + R_3)i_2 = -V_2$$

(iii) Solve to get mesh currents

Note: Branch current, I is different than mesh current, i.

- Example:

Using mesh analysis, find I_0 in the circuit as shown in Figure 3.13

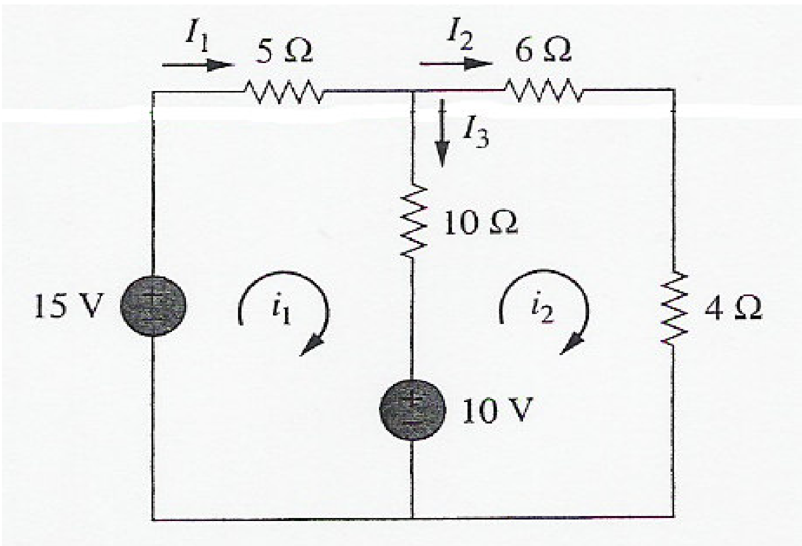

Figure 3.13

For mesh 1, applying KVL,

$$-15+5i_1+10(i_1-i_2)+10=0$$

$$3i_1-2i_2=1$$

For mesh 2,

$$6i_2+4i_2+10(i_2-i_1)-10=0$$

$$i_1=2i_2-1$$

Thus,

$$i_2=1\text{A and } i_1=1\text{A}$$

So that,

$$I_1=i_1=1\text{A}, \qquad I_2=i_2=1\text{A}$$

$$I_3=i_1-i_2=0\text{A}$$

- Exercie:

Using the mesh analysis, find I_0 in the circuit in Figure 3.14. (Answer: -5 A)

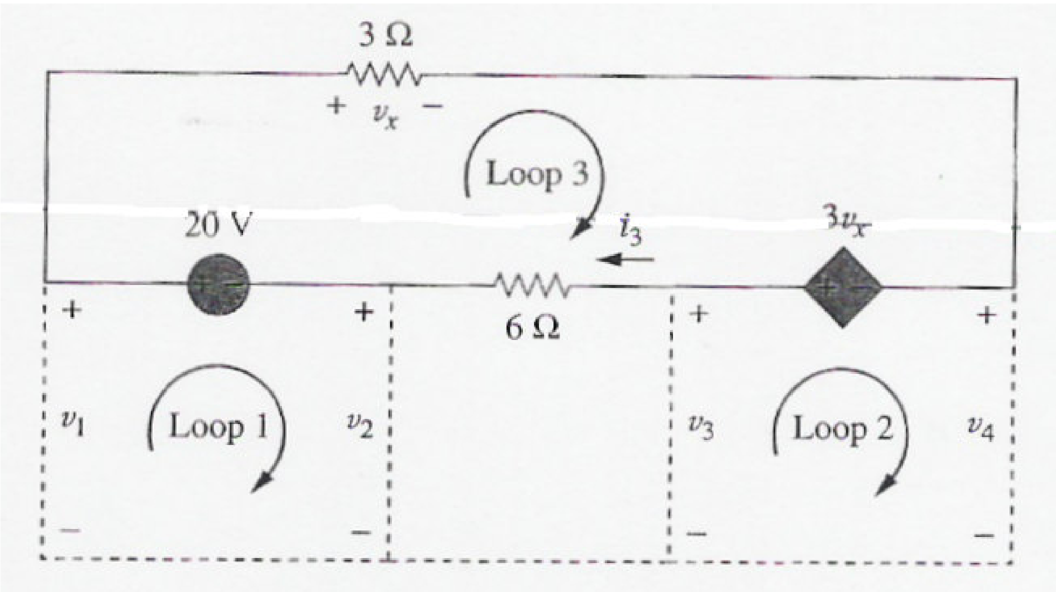

Figure 3.14

3.4 Mesh Analysis with Current Sources

- Applying mesh analysis to circuits containing current sources reduces the number of equations that need to be solved.
- Consider the following two possible cases:

  Case 1:

  When a current source exists only in one mesh.

  Consider the circuit as shown in Figure 3.15.

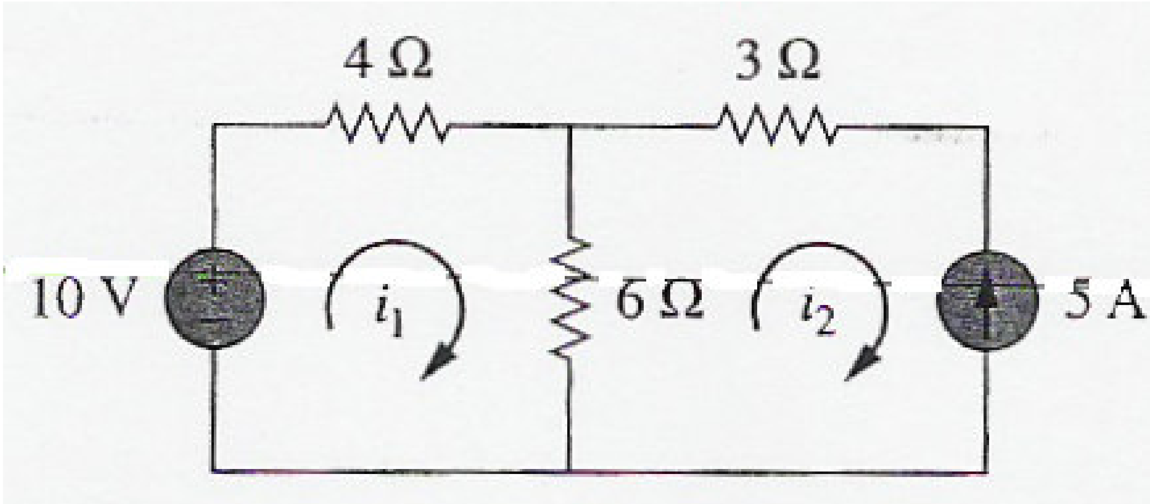

Figure 3.15

We set $i_2 = -5$A and write a mesh equation for the other mesh in the usual way, that is,

Mesh 1: KVL $-10 + 4i_1 + 6(i_1 - i_2) = 0$

Mesh 2: $i_2 = -5A$ $i_1 = -2A$

Case 2:

When a current source exists between two meshes. Consider the circuit as shown in Figure 3.16

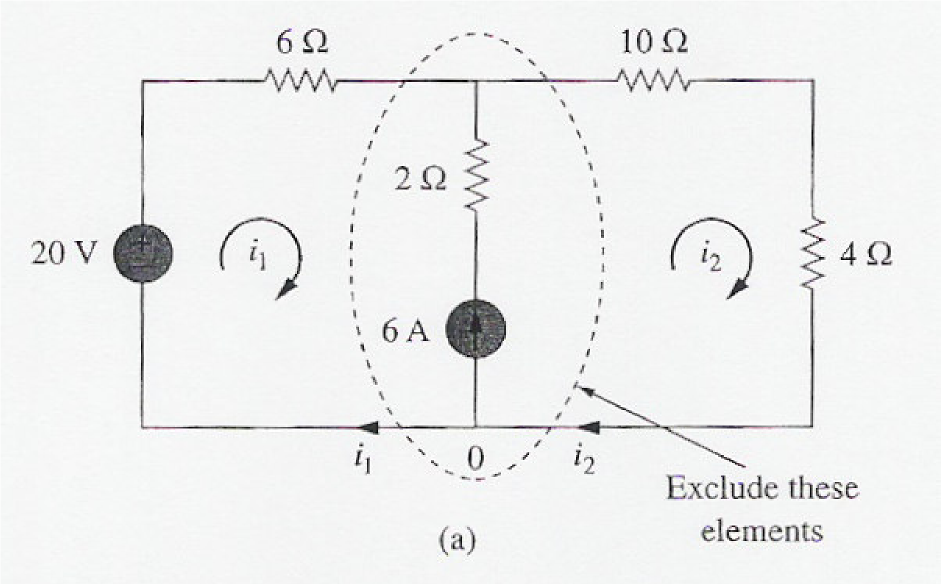

Figure 3.16

For the case, we create supermesh by excluding the current source and any elements connected in series with it as shown in Figure 3.17

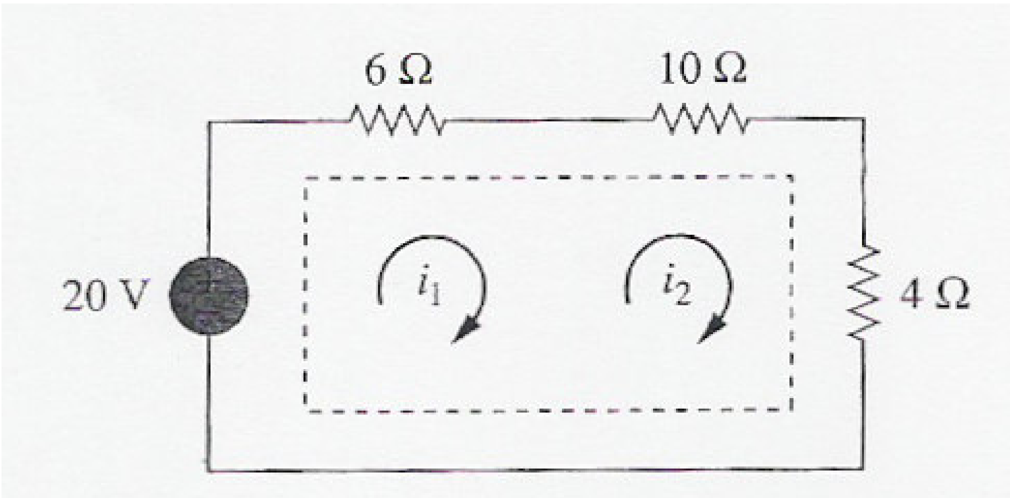

Figure 3.17

KVL:
$$-20+6i_1+10i_2+4i_2=0$$
$$6i_1+14i_2=20$$

KCL at node 0: $i_2=i_1+6$

Solving the simultaneous equations above gives:
$$i_1=-3.2A$$
$$i_2=2.8A$$

- Example:
  Use mesh analysis to determine i_1 to i_4.

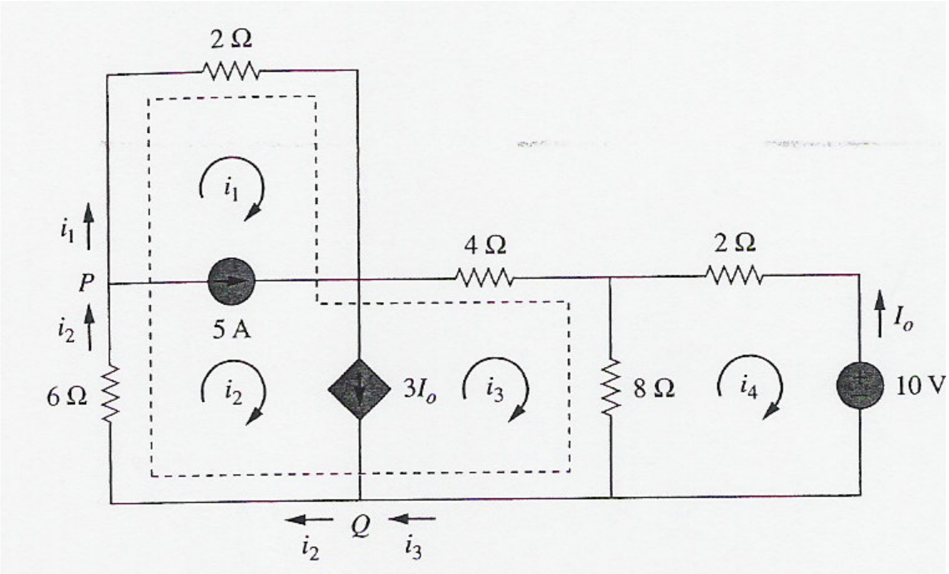

Figure 3.18

Meshes 1 and 2 form a supermesh since they have an independent current source in common.

Meshes 2 and 3 form another supermesh because they have a dependent current source in common.

The two supermeshes intersect and form a larger supermesh as shown.

Applying KVL to the larger supermesh,

$$2i_1 + 4i_3 + 8(i_3 - i_4) + 6i_2 = 0$$

$$i_1 + 3i_2 + 6i_3 - 4i_4 = 0$$

For the independent current source, we apply KCL to node P:

$$i_2 = i_1 + 5$$

For the dependent current source, we apply KCL to node Q:

$$i_2 = i_3 + 3i_0$$

But,

$$i_0 = -i_4$$

Thus,

$$i_2 = i_3 - 3i_4$$

Applying KVL in mesh 4,

$$2i_4 + 8(i_4 - i_3) + 10 = 0$$

$$5i_4 - 4i_3 = -5$$

Therefore,

$$i_1 = -7.5A, \qquad i_2 = -2.5A, \qquad i_3 = 3.93A$$

$$i_4 = 2.143A$$

3.5 Nodal versus Mesh Analysis

- Both nodal and mesh analysis provide a systematic way of analyzing a complex network.
- When we are given a network to be analyzed, how do we know which method is better or suitable? There are 2 factors to help us to decide:

  Factor 1:

  Select the method that results in smaller number of equations.

  Series connected → mesh analysis.

  Parallel connected → nodal analysis.

  Fewer nodes than meshes → nodal analysis.

  Fewer meshes than nodes → mesh analysis.

  Factor 2:

  Information required.

  Node voltages are required → nodal analysis.

  Branh or mesh currents → mesh analysis.